JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **56**, 165–171 (1976)

Periodic Solutions of Certain Differential Equations with Quasibounded Nonlinearities

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Submitted by W. F. Ames

1. INTRODUCTION

Consider the following nonlinear ordinary differential equation:

$$Lx + f(t, x, x', \dots, x^{(n-1)}) = p(t), \tag{1}$$

where L is an nth order linear differential operator with constant coefficients, f and p are continuous functions and periodic in t with period $T > 0$, and $\int_0^T p(u)\, du = 0$. The operator L is assumed to have the following decomposition:

$$L = D^m(D + a_1) \cdots (D + a_{n-m}), \tag{2}$$

where $D = d/dt$, $1 \leqslant m \leqslant n$, and for each $j = 1, \dots, n - m$, $a_j \neq i2\omega\pi/T$ for any integer ω. The function f in (1) is quasibounded [6]. We may define quasiboundedness as follows. Let us denote the norm of each $y = (y_1, \dots, y_n)$ in R^n by $\| y \| = \max_{j=1,\dots,n} | y_j |$. A function $f\colon R^{n+1} \to R$, $(t, y) \to f(t, y)$, which is T-periodic in t, is said to be *quasibounded* if the number

$$| f | = \min_{0<\rho<\infty} \left(\max_{\substack{\|y\| \geqslant \rho \\ 0 \leqslant t \leqslant T}} \frac{| f(t, y)|}{\| y \|} \right) \tag{3}$$

is finite. When f is quasibounded, we call $| f |$ the *quasinorm* of f.

In this paper we shall prove the existence of T-periodic solutions for Eq. (1) by requiring f to be quasibounded and have a quasinorm smaller than certain positive number. This generalizes the results on existence of periodic solutions in Lazer [8] and Chang [2, 3]. Also, we shall prove a uniqueness theorem for the second order equation

$$x'' + cx' + f(t, x) = p(t), \tag{4}$$

which generalizes the uniqueness theorem of Leach [9, Theorem 2.1].

The proof of the existence theorem is based on Schauder's fixed point theorem [10] and on alternative methods (see e.g., Cesari [1], Hale [7]), and uses a technique generalizing those used in Lazer [8] and Chang [2, 3]. The proof of the uniqueness theorem for Eq. (4) is based on Sturm's comparison theorem. We shall state all results in Section 2 and prove them in Section 3.

2. The Results

Let X denote the Banach space of continuous T-periodic functions with the supremum norm, i.e. for each $x \in X$, $\| x \|_X = \max_{0 \leqslant t \leqslant T} | x(t) |$. For each $x \in X$ satisfying $\int_0^T x(u)\, du = 0$, define

$$A(x)(t) = \int_0^t x(u)\, du - \frac{1}{T} \int_0^T \int_0^s x(u)\, du\, ds. \tag{5}$$

It is easy to see that $A(x) \in X$, $\int_0^T A(x)(u)\, du = 0$, $A(x)'(t) = x(t)$, and $\| A(x) \|_X \leqslant T \| x \|_X$. Also, for each $x \in X$ satisfying $\int_0^T x(u)\, du = 0$, define

$$B(x)(t) = \int_0^t x(u)\, du. \tag{6}$$

Then $B(x) \in X$, $B(x)'(t) = x(t)$, and $\| B(x) \|_X \leqslant \frac{1}{2} T \| x \|_X$. For any positive integer k, let A^k denote the operator defined by $A^k(x) = A(\cdots A(x) \cdots)$, repeating k times.

Now, consider the operator L defined by (2). For any $x \in X$, define

$$H_j(x)(t) = (e^{a_j T} - 1)^{-1} \int_t^{T+t} e^{a_j (s-t)} x(s)\, ds, \tag{7}$$

where $j = 1, \ldots, n - m$. It can be shown easily that $H_j(x) \in X$, $(D + a_j) H_j(x) = x$, and

$$\| H_j(x) \|_X \leqslant (1/\rho_j) \| x \|_X, \qquad j = 1, \ldots, n - m, \tag{8}$$

where $\rho_j = | \mathrm{Re}(a_j) |$ when $\mathrm{Re}(a_j) \neq 0$ and $\rho_j = (1/T) | e^{a_j T} - 1 |$ when $\mathrm{Re}(a_j) = 0$ (note that $a_j \neq i 2 \omega \pi / T$).

Let S be the space of all x in X such that x is of class C^{n-1}. Then S becomes a Banach space under the norm

$$\| x \|_S = \max_{j=0,1,\ldots,n-1} \| x^{(j)} \|_X, \qquad \text{where} \qquad x^{(j)} = d^j x / dt^j.$$

Then, for each $x \in X$ satisfying $\int_0^T x(u)\, du = 0$, $H_1 \cdots H_{n-m} B A^{m-1}(x) \in S$ and one can show that

$$\| H_1 \cdots H_{n-m} B A^{m-1}(x) \|_S \leqslant \sigma \| x \|_X, \tag{9}$$

where σ is a positive constant depending on n, m, $a_1,\ldots, a_{n-m}$, $\rho_1,\ldots, \rho_{n-m}$, and T. For example, when Eq. (1) is of the form

$$x^{(n)} + f(t, x, x',\ldots, x^{(n-1)}) = p(t),$$

we consider, instead of the operator $H_1 \cdots H_{n-m}BA^{m-1}$, the operator BA^{n-1} and find that $\| BA^{n-1}(x)\|_S \leqslant \sigma \| x \|_X$ with $\sigma = T^n/2$ when $T > 2$, $\sigma = T^{n-1}$ when $1 \leqslant T \leqslant 2$, and $\sigma = T$ when $0 < T < 1$. Also, when $f = f(t, x)$ in Eq. (1), we simply choose $S = X$ and find that (9) holds with $\sigma = T^m/(2\rho_1 \cdots \rho_{n-m})$.

Now, let us make the following assumptions:

(i) $f: R^{n+1} \to R$ is continuous, $p: R \to R$ is continuous, f and p are periodic in t with period $T > 0$, and $\int_0^T p(u)\, du = 0$.

(ii) There exists a number $M \geqslant 0$ such that for $x \in S$ one of the following two conditions holds:

(a) $\int_0^T f(s, x(s), x'(s),\ldots, x^{(n-1)}(s))\, ds \geqslant 0$ if $x(t) \geqslant M$ for all t, and $\int_0^T f(s, x(s), x'(s),\ldots, x^{(n-1)}(s))\, ds \leqslant 0$ if $x(t) \leqslant -M$ for all t.

(b) Integral in (a) $\leqslant 0$ if $x(t) \geqslant M$ for all t, and $\geqslant 0$ if $x(t) \leqslant -M$ for all t.

(iii) The function f is quasibounded as defined in (3) and has a quasinorm $| f | < \min\{\frac{1}{3}, 1/(6\sigma)\}$.

THEOREM 1. *If the assumptions* (i), (ii), *and* (iii) *are satisfied, then Eq.* (1), *with L given by* (2), *has at least one T-periodic solution.*

COROLLARY. *If we assume* (i, ii), *and*

(iv) $\lim\limits_{\|y\|\to\infty} \dfrac{| f(t, y)|}{\| y \|} = 0$ *uniformly in* t,

then Eq. (1), *with L given by* (2), *has at least one T-periodic solution.*

Remark. Theorem 1 extends the result in [3, Theorem 1] concerning the equation $x^{(n)} + f(t, x) = p(t)$. The above corollary extends the results in Lazer [8] and [2] concerning second order cases.

Consider the second order equation

$$x'' + cx' + f(t, x) = p(t), \tag{4}$$

where c is any real number. Let us make the following assumptions:

(v) $f(t, x)$ and $(\partial/\partial x) f(t, x)$ are continuous for all (t, x), $p(t)$ is continuous for all t, and $f(t, x)$ and $p(t)$ are T-periodic in t.

(vi) $f(t, x)$ satisfies one of the following conditions:

(a) $(\partial/\partial x) f(t, x) < 0$ for all (t, x).

(b) $0 < (\partial/\partial x) f(t, x) < \frac{1}{4}c^2 + (2\pi/T)^2$ for all (t, x).

(c) $\frac{1}{4}c^2 + (2n\pi/T)^2 < (\partial/\partial x) f(t, x) < \frac{1}{4}c^2 + (2(n+1)\pi/T)^2$ for all (t, x) and for some positive integer n.

THEOREM 2. *If the assumptions* (v) *and* (vi) *are satisfied, then Eq.* (4) *has at most one T-periodic solution.*

Remark. Theorem 2 extends the uniqueness theorem of Leach [9, Theorem 2.1] concerning the equation $x'' + f(x) = p(t)$.

3. THE PROOFS

Proof of Theorem 1. For $x \in S$, define

$$F(x)(t) = f(t, x(t), x'(t),\ldots, x^{(n-1)}(t)) - \frac{1}{T}\int_0^T f(s, x(s), x'(s),\ldots, x^{(n-1)}(s))\, ds. \tag{10}$$

Then $F(x) \in X$ and $\int_0^T F(x)(u)\, du = 0$. Let R denote the set of real numbers. For any $\lambda_j \in R$ and $(x_j, r_j) \in S \times R$, $j = 1, 2$, let

$$\lambda_1(x_1, r_1) + \lambda_2(x_2, r_2) = (\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 r_1 + \lambda_2 r_2).$$

Also, for each $(x, r) \in S \times R$, define $|(x, r)| = \| x \|_S + | r |$. Then $S \times R$ becomes a Banach space.

Suppose the condition (ii)-(a) holds. Define a mapping $P: S \times R \to S \times R$ by $P(x, r) = (\tilde{x}, \tilde{r})$ with

$$\tilde{x} = r + H_1 \cdots H_{n-m} B A^{m-1}(p - F(x)), \tag{11}$$

$$\tilde{r} = r - \frac{1}{T}\int_0^T f(s, \tilde{x}(s), \tilde{x}'(s),\ldots, \tilde{x}^{(n-1)}(s))\, ds. \tag{12}$$

It is clear that P is well-defined.

Since the quasinorm $| f | < \min\{\frac{1}{3}, 1/(6\sigma)\}$, there exists $\epsilon > 0$ such that $| f | + \epsilon < \min\{\frac{1}{3}, 1/(6\sigma)\}$. By the definition of quasiboundedness (3), there exists $\rho(\epsilon) > 0$ such that

$$| f(t, y)|/\| y \| < | f | + \epsilon \qquad \text{whenever} \qquad \| y \| \geqslant \rho(\epsilon) \quad \text{and} \quad 0 \leqslant t \leqslant T,$$

where $y = (y_1, \dots, y_n)$. Let

$$L = \max\{|f(t, y)| \mid 0 \leqslant t \leqslant T, \|y\| \leqslant \rho(\epsilon)\},$$

$$N = \max\left\{\frac{M}{1 - 3(|f| + \epsilon)}, \frac{M + 3\sigma\|p\|_X}{1 - 6\sigma(|f| + \epsilon)}, \frac{L}{|f| + \epsilon}, \rho(\epsilon)\right\},$$

and

$$C = \max\{(|f| + \epsilon)\, N, \sigma\|p\|_X + 2\sigma(|f| + \epsilon)\, N\}.$$

We note that $M + 3C \leqslant N$ and

$$|f(t, y)| \leqslant (|f| + \epsilon)\, N \qquad \text{whenever} \qquad \|y\| \leqslant N \quad \text{and} \quad 0 \leqslant t \leqslant T.$$

Now let

$$D = \{(x, r) \in S \times R \mid \|x\|_S \leqslant N, |r| \leqslant M + 2C\}.$$

Then D is a closed, bounded, and convex set in $S \times R$. It can be shown easily that $P(D) \subset D$, P continuous, and $P(D)$ relatively compact. Then by Schauder's fixed point theorem ([10], or see [5, p. 131]) there exists $(\theta, b) \in D$ such that $(\theta, b) = P(\theta, b) = (\tilde{\theta}, \tilde{b})$. From (11) and (12) we see that

$$\frac{1}{T}\int_0^T f(s, \theta(s), \theta'(s), \dots, \theta^{(n-1)}(s))\, ds = 0$$

and hence by (10)

$$F(\theta)\,(t) = f(t, \theta(t), \theta'(t), \dots, \theta^{(n-1)}(t)).$$

Then it follows from $\theta - b = H_1 \cdots H_{n-m} B A^{m-1}(p - F(\theta))$ that

$$D^m(D + a_1) \cdots (D + a_{n-m})\, \theta(t) + f(t, \theta(t), \theta'(t), \dots, \theta^{(n-1)}(t)) = p(t).$$

If the condition (ii)-(b) holds, we redefine the $\tilde{r}$ in (12) as

$$\tilde{r} = r + \frac{1}{T}\int_0^T f(s, \tilde{x}(s), \tilde{x}'(s), \dots, \tilde{x}^{(n-1)}(s))\, ds.$$

Then a similar argument as before establishes the desired result.

Proof of Corollary. The condition (iv) implies that $|f| = 0$.

Proof of Theorem 2. Suppose on the contrary that there exist two distinct T-periodic solutions $x_1(t)$ and $x_2(t)$ to Eq. (4). Then $\varphi(t) = x_2(t) - x_1(t)$ is a nontrivial T-periodic solution of

$$y'' + cy' + q(t)\, y = 0, \tag{13}$$

where

$$q(t) = \int_0^1 \frac{\partial}{\partial x} f[t, x_1(t) + s(x_2(t) - x_1(t))]\, ds.$$

Note that Eq. (13) is equivalent to

$$(e^{ct}y')' + e^{ct}q(t)\,y = 0. \tag{14}$$

Suppose the condition (vi)-(a) holds. Then $q(t) < 0$ and hence $e^{ct}q(t) < 0$. We claim that there exists t_0 such that $\varphi(t_0) = 0$ and $\varphi'(t_0) \neq 0$. For if $\varphi(t) > 0$ for all t, we have $(e^{ct}\varphi'(t))' = -e^{ct}q(t)\,\varphi(t) > 0$ and hence $e^{ct}\varphi'(t)$ strictly increasing, which is impossible for the nontrivial T-periodic solution $\varphi(t)$. Similarly it is impossible to have $\varphi(t) < 0$ for all t. Thus we have $\varphi(t_0) = \varphi(t_0 + T) = \cdots = 0$. But, comparing Eq. (14) with the equation $(e^{ct}y')' = 0$, we obtain immediately a contradiction by Sturm's comparison Theorem (see e.g., [4, p. 208]).

Since the proof for the case (vi)-(b) is similar to both (vi)-(a) and (vi)-(c), we shall only prove (vi)-(c) in the following.

Suppose the condition (vi)-(c) holds. Then we have

$$e^{ct}[\tfrac{1}{4}c^2 + (2n\pi/T)^2] < e^{ct}q(t) < e^{ct}[\tfrac{1}{4}c^2 + (2(n+1)\,\pi/T)^2].$$

Comparing Eq. (14) with the equation

$$(e^{ct}y')' + e^{ct}[\tfrac{1}{4}c^2 + (2n\pi/T)^2]\,y = 0, \tag{15}$$

we conclude immediately that there exists t_0 such that $\varphi(t_0) = 0$ and $\varphi'(t_0) \neq 0$. Now, since Eq. (15) has a solution

$$y(t) = \exp(-\tfrac{1}{2}ct)\sin[2n\pi(t - t_0)/T]$$

with $2n + 1$ zeros on $[t_0, t_0 + T]$, $\varphi(t)$ has at least $2n$ zeros in the open interval $(t_0, t_0 + T)$ by Sturm's comparison theorem. On the other hand, since the equation

$$(e^{ct}y')' + e^{ct}[\tfrac{1}{4}c^2 + (2(n+1)\,\pi/T)^2]\,y = 0$$

has a solution $y(t) = \exp(-\tfrac{1}{2}ct)\sin[2(n+1)\,\pi(t - t_0)/T]$ with $2n + 1$ zeros in $(t_0, t_0 + T)$ and since $\varphi(t_0) = \varphi(t_0 + T) = 0$, we conclude that $\varphi(t)$ can have at most $2n$ zeros in $(t_0, t_0 + T)$. Hence, $\varphi(t)$ has exactly $2n$ zeros in $(t_0, t_0 + T)$. This implies that $\varphi'(t_0)$ and $\varphi'(t_0 + T)$ have opposite signs, a contradiction to that φ is a T-periodic solution. This completes the proof.

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